Problem 3.17

Solve Equation 3.69 for $\Psi(x)$. Note that $\langle x\rangle$ and $\langle p\rangle$ are *constants* (independent of x).

Solution

Equation 3.69 is the differential equation for the minimum-uncertainty wave packet.

$$\left(-i\hbar\frac{d}{dx} - \langle p\rangle\right)\Psi = ia(x - \langle x\rangle)\Psi \tag{3.69}$$

Expand both sides.

$$-i\hbar\frac{d\Psi}{dx} - \langle p\rangle\Psi = iax\Psi - ia\langle x\rangle\Psi$$

Bring all terms to the left side and factor Ψ.

$$-i\hbar\frac{d\Psi}{dx} + (-iax + ia\langle x\rangle - \langle p\rangle)\Psi = 0$$

Multiply both sides by i and divide both sides by $\hbar$.

$$\frac{d\Psi}{dx} + \left(\frac{a}{\hbar}x - \frac{a\langle x\rangle}{\hbar} - \frac{i\langle p\rangle}{\hbar}\right)\Psi = 0 \tag{1}$$

This is a first-order linear ODE for Ψ, so it can be solved with an integrating factor I.

$$I = \exp\left[\int^x \left(\frac{a}{\hbar}r - \frac{a\langle x\rangle}{\hbar} - \frac{i\langle p\rangle}{\hbar}\right)dr\right] = \exp\left(\frac{a}{2\hbar}x^2 - \frac{a\langle x\rangle}{\hbar}x - \frac{i\langle p\rangle}{\hbar}x\right)$$

Multiply both sides of equation (1) by I.

$$\exp\left(\frac{a}{2\hbar}x^2 - \frac{a\langle x\rangle}{\hbar}x - \frac{i\langle p\rangle}{\hbar}x\right)\frac{d\Psi}{dx} + \left(\frac{a}{\hbar}x - \frac{a\langle x\rangle}{\hbar} - \frac{i\langle p\rangle}{\hbar}\right)\exp\left(\frac{a}{2\hbar}x^2 - \frac{a\langle x\rangle}{\hbar}x - \frac{i\langle p\rangle}{\hbar}x\right)\Psi = 0$$

The left side can be written as $d/dx(I\Psi)$ by the product rule.

$$\frac{d}{dx}\left[\exp\left(\frac{a}{2\hbar}x^2 - \frac{a\langle x\rangle}{\hbar}x - \frac{i\langle p\rangle}{\hbar}x\right)\Psi\right] = 0$$

Integrate both sides with respect to x.

$$\exp\left(\frac{a}{2\hbar}x^2 - \frac{a\langle x\rangle}{\hbar}x - \frac{i\langle p\rangle}{\hbar}x\right)\Psi = C$$

Solve for $\Psi(x)$.

$$\begin{aligned}\Psi(x) &= C\exp\left(-\frac{a}{2\hbar}x^2 + \frac{a\langle x\rangle}{\hbar}x + \frac{i\langle p\rangle}{\hbar}x\right) \\ &= C\exp\left(-\frac{a}{2\hbar}x^2 + \frac{a\langle x\rangle}{\hbar}x\right)\exp\left(\frac{i\langle p\rangle}{\hbar}x\right)\end{aligned}$$

Complete the square in the exponent.

$$\Psi(x) = C \exp\left[-\frac{a}{2\hbar}(x^2 - 2\langle x\rangle x)\right] \exp\left(\frac{i\langle p\rangle}{\hbar}x\right)$$

$$= C \exp\left(\frac{a}{2\hbar}\langle x\rangle^2\right) \exp\left[-\frac{a}{2\hbar}(x^2 - 2\langle x\rangle x + \langle x\rangle^2)\right] \exp\left(\frac{i\langle p\rangle}{\hbar}x\right)$$

Therefore, letting A be the arbitrary constant,

$$\Psi(x) = A \exp\left[-\frac{a}{2\hbar}(x - \langle x\rangle)^2\right] \exp\left(\frac{i\langle p\rangle}{\hbar}x\right);$$

the minimum-uncertainty wave packet is a gaussian function.